

Nonlocal and controlled unitary operators of Schmidt rank three

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Abstract

Implementing nonlocal unitary operators is an important and hard question in quantum computing and cryptography. We show that any bipartite nonlocal unitary operator of Schmidt rank 3 on the $(d_A \times d_B)$ -dimensional system is locally equivalent to a controlled unitary when d_A is at most 3. This operator can be locally implemented assisted by a maximally entangled state of Schmidt rank $r = \min\{d_A^2, d_B\}$. We further show that stochastic-equivalent nonlocal unitary operators are indeed locally equivalent, and propose a sufficient condition on which nonlocal and controlled unitary operators are locally equivalent. We also provide the solution to a special case of a conjecture on the ranks of multipartite quantum states.

1 Introduction

Implementing multipartite unitary operators is a fundamental task in quantum information theory. The operators are called *local* when they are the tensor product of unitary operators locally acting on subsystems, i.e., they have *Schmidt rank* one. Otherwise they are called *nonlocal*. It is known that the local unitary can be implemented by local operations and classical communication (LOCC) with probability one. Recent research has been devoted to the decomposition of local unitaries into elementary operations [1], and the local equivalence between multipartite quantum states of fermionic systems under local unitaries [2, 3, 4].

Nonlocal unitary operators have a more complex structure and play a more powerful role than local unitaries in quantum computing, cryptography and so on. Nonlocal unitaries can create quantum entanglement between distributed parties [5], and their equivalence has been studied under LOCC [6]. So nonlocal unitaries cannot be implemented by LOCC only, even if the probability is allowed to be close to zero [7]. The understanding of the forms and implementation schemes of nonlocal unitary operators is still far from complete. A simplest type of nonlocal unitaries is the *controlled unitary* gates, which are of the general form $U = \sum_{j=1}^m P_j \otimes V_j$ acting on a bipartite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, where P_j are orthogonal projectors on \mathcal{H}_A and V_j are unitaries on \mathcal{H}_B . They can be implemented by a simple nonlocal protocol [8] using a maximally entangled state of Schmidt rank m . Some other types of nonlocal unitaries are discussed in [8], but in this paper we will focus on controlled unitaries. (Note that entirely different implementations are possible if the systems are deemed to be near enough so as to allow direct quantum interactions between them, e.g. multiqubit controlled gates can be decomposed into certain elementary gates [9]). Recently an interesting connection between nonlocal and controlled unitaries was found: they are locally equivalent when they have Schmidt rank 2 [10]. In this case their implementations are the same

and operational. So it is important to strengthen this connection for operationally implementing more nonlocal unitaries.

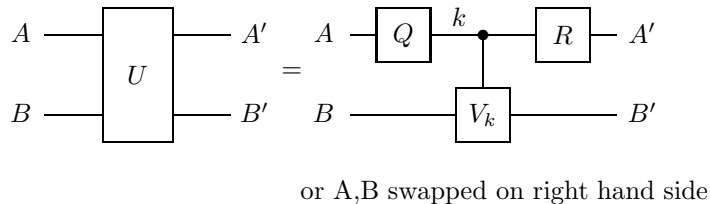


Figure 1: Any bipartite unitary U on $d_A \times d_B$ system of Schmidt rank 3 is locally equivalent to a controlled unitary when $d_A = 2, 3$, where the controlling side may be A or B . This is expressed as $U = (Q \otimes I)(\sum_{k=1}^{d_A} |k\rangle\langle k| \otimes V_k)(R \otimes I)$ or $U = (I \otimes Q)(\sum_{k=1}^{d_B} V_k \otimes |k\rangle\langle k|)(I \otimes R)$, where V_k , Q and R are local unitaries. The output systems A' and B' are assumed to be of the same size as A and B , respectively.

In this paper we show that any bipartite unitary operation U of Schmidt rank 3 on $d_A \times d_B$ system is locally equivalent to a controlled unitary when $d_A = 2, 3$, see Theorems 3 and 6. This is illustrated in Fig. 1. They not only imply the method of implementing U but also simplify the structure of U . We also propose an operational method of explicitly decomposing U into the form of controlled unitaries in the end of Sec. 3. As an application we can simplify the problem of deciding the SL-equivalence of two bipartite unitaries of Schmidt rank 3 with $d_A = 2, 3$. This is based on Theorem 7 that any two SL-equivalent nonlocal unitary operators are locally equivalent. Using this theorem we provide a sufficient condition by which a bipartite unitary is locally equivalent to a controlled unitary in Corollary 8. Next we show that U can be implemented by LOCC and a maximally entangled state $|\Psi_r\rangle = \frac{1}{\sqrt{r}} \sum_{i=1}^r |ii\rangle$, where $r = \min\{d_A^2, d_B\}$ in Lemma 9. Next, we apply our result to solve a special case of a conjecture on the ranks of multipartite quantum states, see Conjecture 10.

Controlled unitary operators are one of the most easily accessible and extensively studied quantum operators. For example, the controlled NOT (CNOT) gate is essential to construct the universal quantum two-qubit gate used in quantum computing [9]. Experimental schemes of implementing the CNOT gates have also been proposed, such as cavity QED technique [11] and trapped ions [12]. Recently CNOT gates have been proved to be decomposed in terms of a two-qubit entangled gate and single qubit phase gates, which could be implemented by trapped ions controlled by fully overlapping laser pulses [13]. Next, multiqubit graph states for one-way quantum computing are generated by a series of controlled-Z gates [14]. Third, controlled phase gates have been used to construct the mutually unbiased bases (MUBs) [15] and graph states for which the violation of multipartite Bell-type inequalities have been experimentally demonstrated [16]. These applications (and those not mentioned above) could be improved by the strengthened connection between nonlocal and controlled unitaries presented in this paper.

The rest of this paper is organized as follows. In Sec. 2 we introduce the preliminary knowledge, and propose Conjecture 1 as the main question in this paper. In Sec. 3 we prove Conjecture 1 when $d_A = 2, 3$, and we propose its applications on general nonlocal unitaries in Sec. 4. Finally we conclude in Sec. 5.

2 Preliminaries

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the complex Hilbert space of a finite-dimensional bipartite quantum system of Alice and Bob. We denote by d_A, d_B the dimension of \mathcal{H}_A and \mathcal{H}_B , respectively. It is known

that \mathcal{H} is spanned by the computational basis $|i, j\rangle, i = 1, \dots, d_A, j = 1, \dots, d_B$. We shall denote $I_k = \sum_{i=1}^k |i\rangle\langle i|$. For convenience, we denote $I_A = I_{d_A}$, $I_B = I_{d_B}$ and $I = I_{d_A d_B}$ as the identity operator on spaces $\mathcal{H}_A, \mathcal{H}_B$, and \mathcal{H} , respectively. Let $d = d_A d_B$, and $U, V \in \text{U}(d)$ be two unitary matrices on the space \mathcal{H} . We say that U, V are equivalent under stochastic local operations, or *SL-equivalent* when there are two locally invertible matrices $S_1, S_2 \in \text{GL}(d_A) \times \text{GL}(d_B)$ such that $U = S_1 V S_2$. In particular, we say that U, V are *locally equivalent* when S_1, S_2 are unitary.

A unitary matrix U on \mathcal{H} has Schmidt rank n if there is a decomposition $U = \sum_{j=1}^n A_j \otimes B_j$ where the $d_A \times d_A$ matrices A_1, \dots, A_n are linearly independent, and the $d_B \times d_B$ matrices B_1, \dots, B_n are linearly independent. We say that U is a *controlled unitary gate*, if U is locally equivalent to $\sum_{j=1}^{d_A} |j\rangle\langle j| \otimes U_j$ or $\sum_{j=1}^{d_B} V_j \otimes |j\rangle\langle j|$. To be specific, U is a controlled unitary from A or B side, respectively. Clearly the matrices U_j, V_j are unitary. We further say that system A (or B) controls with n terms if $U = \sum_{i=1}^n P_i \otimes U_i$ (or $\sum_{i=1}^n U_i \otimes P_i$), where the U_1, \dots, U_n are linearly independent unitaries and the P_i are orthogonal projectors, i.e., $P_i P_j = \delta_{ij} P_i$.

It is known that any multipartite (i.e., nonlocal) unitary gate of Schmidt rank 2 is a controlled unitary [10]. However a bipartite unitary of Schmidt rank 4 may be not a controlled unitary, say the two-qubit SWAP gate [10]. It is then an interesting question to characterize the bipartite unitary of Schmidt rank 3. Formally, we investigate the following conjecture in the next section.

Conjecture 1 *Any bipartite unitary operator of Schmidt rank 3 is a controlled unitary operator.*

To approach this conjecture, we generalize the concept of controlled unitary gate. We split the space into a direct sum: $\mathcal{H}_A = \oplus_{i=1}^m \mathcal{H}_i$, $m > 1$, $\text{Dim } \mathcal{H}_i = m_i$, and $\mathcal{H}_i \perp \mathcal{H}_j$ for distinct $i, j = 1, \dots, m$. We say that U is a *block-controlled unitary (BCU) gate controlled from the A side*, if U is locally equivalent to $\sum_{i=1}^m \sum_{j,k=1}^{m_i} |u_{ij}\rangle\langle u_{ik}| \otimes U_{ijk}$ where $|u_{ij}\rangle \in \mathcal{H}_i$ for $j, k = 1, \dots, m_i$. For simplicity we denote the decomposition as $\oplus_A V_i$ where $V_i = \sum_{j,k=1}^{m_i} |u_{ij}\rangle\langle u_{ik}| \otimes U_{ijk}$, and we denote $|V_i|_A = m_i$. We have $U U^\dagger = \sum_{i=1}^m P_i \otimes I_B = I$, where P_i is the projector on the space \mathcal{H}_i . So the BCU from the A side can be understood as the direct sum of nonlocal unitaries on the spaces $\mathcal{H}_i \otimes \mathcal{H}_B$, $i = 1, \dots, m$. In particular if $m_i = 1$ for all i , then U degenerates to a controlled unitary gate from the A side. So a BCU has more general properties than those a controlled unitary gate has. One may similarly define the BCU gate controlled from the B side.

Although the controlled unitary gate is a BCU gate, the converse is wrong. An example is the following qutrit-qubit unitary gate: $U = \frac{1}{2}(I_2 \otimes I_2 + \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) + |3\rangle\langle 3| \otimes I_2$, where $\sigma_x, \sigma_y, \sigma_z$ are the standard Pauli operators. By definition U is a BCU gate from A side. If U is a controlled unitary from A or B side, then it has Schmidt rank at most 3 or 2. It is a contradiction with the fact that U has Schmidt rank four. So U is not a controlled unitary.

Since a nonlocal unitary and controlled unitary may be not locally equivalent, one may ask when they are locally equivalent. The question has been addressed in [10, Lemma 2]. However the lemma is not very operational in practice. Below we present an operational criterion based on [17, Corollary 5] and [10, Lemma 2]. Note that [18] also cited some related work of the authors of [17], in studying the entanglement cost of more general types of bipartite unitaries.

Lemma 2 *Let $U = \sum_{j=1}^r A_j \otimes B_j$ be a nonlocal unitary of Schmidt rank r . Then U is a controlled unitary from A side if and only if $A_i A_j^\dagger$, $i, j = 1, \dots, r$ are all normal and commute with each other, and $A_i^\dagger A_j$, $i, j = 1, \dots, r$ are all normal and commute with each other.*

3 Proving Conjecture 1 for $d_A = 2, 3$

Conjecture 1 trivially holds for $d_A = 1$. In this section we show that Conjecture 1 holds when one of the systems A, B is a qubit or a qutrit. The first case is demonstrated by the following theorem.

Theorem 3 Any bipartite unitary on $2 \times d_B$ of Schmidt rank 3 is locally equivalent to a controlled unitary controlled from the B side.

Proof. Let U be a bipartite unitary on $2 \times d_B$ of Schmidt rank 3. Suppose U has an operator Schmidt expansion $U = \sum_{j=1}^3 E_j \otimes F_j$. Using the orthogonality under the Hilbert-Schmidt inner product, there is a 2×2 matrix E_4 orthogonal to E_1, E_2, E_3 . Let the nonnegative real numbers a, b be the singular values of E_4 . Up to a local unitary we may assume $E_4 = a|0\rangle\langle 0| + b|1\rangle\langle 1|$. Since U has Schmidt rank 3, U is locally equivalent to the unitary $U_1 = \sum_{j=1}^3 A_j \otimes B_j$ where $A_1 = |0\rangle\langle 1|$, $A_2 = |1\rangle\langle 0|$, $A_3 = b|0\rangle\langle 0| - a|1\rangle\langle 1|$, and B_3 a diagonal matrix.

Let $\mathcal{H}_B = \oplus_{i=1}^k V_i$ be an orthogonal decomposition and the diagonal matrix P_i the projector on the subspace V_i , $\forall i$. So $P_i P_j = \delta_{ij} P_i$ and $\sum_{i=1}^k P_i = I_B$. Up to a local unitary we may assume the orthogonal decomposition $B_3 = \sum_{i=1}^k c_i P_i$, where $c_i > c_j \geq 0$ for all $i < j$. Since U_1 is unitary, we have

$$b^2 B_3^\dagger B_3 + B_2^\dagger B_2 = B_1^\dagger B_1 + a^2 B_3^\dagger B_3 = I_B, \quad (1)$$

$$b^2 B_3 B_3^\dagger + B_1 B_1^\dagger = B_2 B_2^\dagger + a^2 B_3 B_3^\dagger = I_B. \quad (2)$$

Taking the trace in Eqs. (1) and (2), we have $a = b > 0$. Since U_1 is unitary, we have

$$B_3^\dagger B_1 = B_2^\dagger B_3, B_1 B_3^\dagger = B_3 B_2^\dagger. \quad (3)$$

Since $B_3 = B_3^\dagger = \sum_{i=1}^k c_i P_i$, B_1 and B_2 both commute with B_3^2 . Since $c_i > c_j \geq 0$ for all $i < j$, we have $B_1 = \oplus_{i=1}^k G_i$ and $B_2 = \oplus_{i=1}^k H_i$, where the square blocks G_i, H_i act on the space V_i , $\forall i$. By (3) we have $G_1 = H_1^\dagger, \dots, G_{k-1} = H_{k-1}^\dagger$ and $c_k G_k = c_k H_k^\dagger$. It follows from (1) and (2) that

$$B_1^\dagger B_1 = B_1 B_1^\dagger = B_2^\dagger B_2 = B_2 B_2^\dagger = I_B - a^2 B_3 B_3^\dagger. \quad (4)$$

So the matrices G_1, \dots, G_{k-1} and H_1, \dots, H_{k-1} are normal. If $c_k > 0$ then $B_1 = B_2^\dagger$ and G_k is also normal by (4). So B_1 is normal, and B_1, B_2, B_3 are simultaneously diagonalizable under unitary similarity transformation. So U_1 is locally equivalent to a controlled unitary controlled from the B side. Since U, U_1 are locally equivalent, the assertion follows. On the other hand if $c_k = 0$, by (4) we have $G_k G_k^\dagger = H_k H_k^\dagger = P_k$, i.e., both G_k, H_k are unitary. So B_1, B_2, B_3 are simultaneously locally equivalent to diagonal matrices, and the assertion follows. This completes the proof. \square

The controlled unitary on $2 \times d_B$ of Schmidt rank 3 cannot be controlled from A side, otherwise the Schmidt rank would be decreased. Below we construct a controlled unitary on 3×3 of Schmidt rank 3 which is not controlled from A side. Let $U = \sum_{i=1}^3 V_i \otimes |i\rangle\langle i|$, where $V_i = U_i \oplus |3\rangle\langle 3|$, $i = 1, 2, 3$ and the U_i are linearly independent unitaries on the 2×2 system. One can verify that U is a controlled unitary of Schmidt rank 3 controlled from B side. If it is also controlled from A side, then the 3-dimensional subspace H spanned by the V_i is also spanned by three matrices of rank one. This is a contradiction with the fact that there is no matrix of rank one in H . So U is not controlled from A side. Next let $U' = \sum_{i=1}^3 V_i \otimes P_i$ be a controlled unitary on $3 \times d_B$ and B control with three terms. Using a similar argument above, we can show that U' is not controlled from A side.

It is known that [10, Theorem 6] shows two facts. Any bipartite unitary U of Schmidt rank 2 (i) is controlled from both A and B sides, and (ii) has at least one of the two systems A, B controlling with two terms. Can these two statements be generalized to unitaries of Schmidt rank 3? The bipartite unitary in Theorem 3 and U, U' in the last paragraph have Schmidt rank 3 and violates statement (i). Next we show that statement (ii) cannot be generalized to that one side controls with three terms. Consider the controlled unitary $V = I_2 \otimes |1\rangle\langle 1| + \sigma_x \otimes |2\rangle\langle 2| + \sigma_z \otimes |3\rangle\langle 3| + \frac{\sigma_x + \sigma_z}{\sqrt{2}} \otimes |4\rangle\langle 4|$ of Schmidt rank 3 on 2×4 system. Evidently A side cannot control with three terms. If B side controls with three terms, then V is locally equivalent to $V' = \sum_{i=1}^3 U_i \otimes P_i$ where P_i are

pairwise orthogonal projectors. In any expansion of the Schmidt-rank-3 unitary V of the form $V = \sum_{j=1}^3 A_j \otimes B_j$, the subspace $\text{span}\{B_j\}$ is well-defined in the sense that it is determined solely by V and is independent of the form of the expansion (as long as the expansion has only 3 terms), so for this particular V this subspace is the 3-dimensional subspace S_1 spanned by the matrices $|1\rangle\langle 1|, |2\rangle\langle 2| + \frac{1}{\sqrt{2}}|4\rangle\langle 4|, |3\rangle\langle 3| + \frac{1}{\sqrt{2}}|4\rangle\langle 4|$, because we can choose A_j to be I_2, σ_x and σ_z . The corresponding subspace for V' is $\text{span}\{P_i\}$, which contains two linearly independent matrices of rank one. As V and V' are locally equivalent, the subspace S_1 also contains two linearly independent matrices of rank one. This is impossible and hence B side cannot control with three terms. Therefore the statement (ii) cannot be directly generalized to the case of Schmidt rank 3.

To investigate Conjecture 1 with a qutrit system, we present two preliminary lemmas.

Lemma 4 *Assertion (i) below implies assertion (ii):*

- (i) *any bipartite unitary on $d_A \times d_B$ system of Schmidt rank 3 is locally equivalent to a controlled unitary;*
- (ii) *any bipartite BCU from A side on $(d_A + 1) \times d_B$ of Schmidt rank 3 is locally equivalent to a controlled unitary.*

Proof. Let U be a bipartite BCU from A side on $(d_A + 1) \times d_B$ of Schmidt rank 3. We may assume $U = U_1 \oplus_A U_2$. Since U has Schmidt rank 3, U_1, U_2 have Schmidt rank at most 3. Since both $|U_1|_A, |U_2|_A \leq d_A$, it follows from (i) and [10] that both of them are equivalent to locally unitaries. If both of them are controlled from A side, then (ii) holds. If one of them is controlled from only B side, then it has Schmidt rank 3 [10]. Since U also has Schmidt rank 3, it is a controlled unitary from B side. So (ii) holds. This completes the proof. \square

An open problem is whether the converse is true, i.e., (ii) \rightarrow (i).

Lemma 5 *Any bipartite operator on \mathcal{H} of Schmidt rank at most d_A is locally equivalent to another operator $\sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes U_{ij}$ such that $U_{ij} = 0$ for a pair of subscripts (i, j) .*

Proof. Let $V = \sum_{i,j=1}^{d_A} |i\rangle\langle j| \otimes V_{ij}$ be a random bipartite operator of Schmidt rank at most d_A . Using the row and column operations, we only need to show that we can always realize $V_{11} = 0$. First this is evidently true when the blocks $V_{i,1}, \dots, V_{i,d_A}$ are linearly dependent for some i . Next suppose they are linearly independent for $i = 1, \dots, d_A$. Since the Schmidt rank of V is at most d_A , it becomes exactly d_A now. So $V_{1,1}, \dots, V_{1,d_A}$ are in the d_A -dimensional subspace spanned by $V_{2,1}, \dots, V_{2,d_A}$. There is a unit vector (x, y) such that the following d_A matrix pencils $xV_{1,1} + yV_{2,1}, \dots, xV_{1,d_A} + yV_{2,d_A}$ are linearly dependent. Let $(x, y, 0, \dots, 0)$ be the first row of the unitary matrix W . Then the d_A top blocks of size $d_B \times d_B$ in $(W \otimes I)V$ are exactly the above matrix pencils, so they are linearly dependent. Now the claim follows from the first case. This completes the proof. \square

The assertion of this lemma can be easily generalized to the case in which the bipartite operator V is replaced by an isometry mapping the space $\mathbf{C}^n \otimes \mathbf{C}^q$ to $\mathbf{C}^m \otimes \mathbf{C}^p$, i.e., $V = \sum_{i=1}^m \sum_{j=1}^n |i\rangle\langle j| \otimes V_{ij}$ where V_{ij} is of size $p \times q$. Now we are in a position to prove Conjecture 1 with $d_A = 3$. We shall denote $A \propto B$ for two linearly dependent matrices A, B .

Theorem 6 *Any bipartite unitary on $3 \times d_B$ of Schmidt rank 3 is locally equivalent to a controlled unitary.*

Proof. Let U be a bipartite unitary on $3 \times d_B$ of Schmidt rank 3. We use the induction on d_B . For $d_B = 2$ the assertion follows from Theorem 3. Suppose the assertion holds for $2, \dots, d_B - 1$. We claim that the assertion holds when U is a BCU. If U is controlled from A side, then the claim follows from Lemma 4 and Theorem 3. Let U be controlled from B side. We have $U = U_1 \oplus_B U_2$ where the unitaries $U_i \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_i)$, $i = 1, 2$ and $\mathcal{H}_1 \oplus \mathcal{H}_2 = \mathcal{H}_B$. Since U has Schmidt rank

3, U_1, U_2 have Schmidt rank at most 3. Since both $|U_1|_B, |U_2|_B < d_B$, the induction hypothesis and [10] imply that U_1, U_2 are controlled unitaries. If both U_1, U_2 are controlled from the B side then the claim follows. Suppose one of them, say U_1 is controlled from the A side only. So U_1 has Schmidt rank 3 [10]. Since U also has Schmidt rank 3, it is a controlled unitary from the A side, so the claim follows. From now on we assume that U is not a BCU.

By Lemma 5 we may assume the bipartite unitary $U = \sum_{i,j=1}^3 |i\rangle\langle j| \otimes U_{ij}$ with U_{ij} of size $d_B \times d_B$ and $U_{13} = 0$. Since U is unitary, the submatrix $\sum_{i=2}^3 \sum_{j=1}^2 |i\rangle\langle j| \otimes U_{ij}$ has rank at most d_B . There is a unit vector (x, y) such that the matrix pencil $x(U_{21}, U_{22}) + y(U_{31}, U_{32})$ has rank at most $d_B - 1$. Let V_1 be a 3×3 unitary with the bottom row $(0, x, y)$, and $U' = (V_1 \otimes I_B)U$. The submatrix consisting of the bottom left and middle $d_2 \times d_2$ blocks of U' is exactly the above-mentioned matrix pencil. So U' is locally equivalent to $W = \sum_{i,j=1}^3 |i\rangle\langle j| \otimes W_{ij}$, where the bottom row of W is $(0, \dots, 0, 1)$. Since W is not a BCU, the three blocks W_{31}, W_{32}, W_{33} are linearly dependent. It implies $W_{31} \propto W_{32}$. So W is locally equivalent to $W' = \sum_{i,j=1}^3 |i\rangle\langle j| \otimes W'_{ij}$ with $W'_{13} = W'_{31} = 0$. Since W' is unitary, the rank of W'_{23} and W'_{32} are equal. So W' is locally equivalent

to $V = \begin{pmatrix} v_1 & v_2 & 0 \\ v_3 & v_4 & v_5 \\ 0 & v_6 & v_7 \end{pmatrix}$, where the $d_B \times d_B$ blocks v_i have the expression $v_i = \begin{pmatrix} v_{i1} & v_{i2} \\ v_{i3} & v_{i4} \end{pmatrix}$,

$i = 1, \dots, 7$, and $v_{22} = v_{24} = v_{33} = v_{34} = v_{42} = v_{43} = v_{51} = v_{52} = v_{61} = v_{63} = 0$. The blocks v_{i1} are of size $(d_B - r) \times (d_B - r)$, v_{i2} of size $(d_B - r) \times r$, v_{i3} of size $r \times (d_B - r)$, and v_{i4} of size $r \times r$. Since W is not a BCU, none of $v_1, v_2, v_3, v_5, v_6, v_7$ is zero. So $r \in [1, d_B - 1]$, and v_2, v_6 are linearly independent. Let H be the space spanned by v_2, v_3, v_5, v_6 . Since the Schmidt rank of V is 3, we have $\text{Dim } H = 2$ or 3.

Suppose $\text{Dim } H = 2$, so $v_3, v_5 \in H = \text{span}\{v_2, v_6\}$. We have two cases (1) $v_{23} = 0, v_{64} \neq 0$, and (2) $v_{23} \neq 0, v_{64} = 0$. In case (1), we have $v_{32} = v_{53} = v_{62} = 0, v_{21} \propto v_{31}$ and $v_{54} \propto v_{64}$. If $v_4 \in H$, then v_{21} and v_{64} are both invertible. Since V is unitary, we have $v_{12} = v_{13} = v_{72} = v_{73} = 0$. Then V becomes a BCU which gives us a contradiction with the assumption. So $v_4 \notin H$. The space spanned by the v_i is spanned by v_3, v_4, v_5 . Again V becomes a BCU and we have a contradiction. In case (2), we have $v_{31} = v_{21} = v_{54} = 0, v_{32} \propto v_{62}$ and $v_{23} \propto v_{53}$. If $v_4 \in H$, then $v_4 = 0$. Since V is unitary, we have $r = d_B - r$ and v_{23}, v_{32} are invertible. It implies that V has Schmidt rank 4 and we have a contradiction. So $v_4 \notin H$. Since V has Schmidt rank 3, we have $v_{i3} \propto v_{23}$ and $v_{i2} \propto v_{32}$. Since V is unitary, we have $v_{12} = v_{13} = v_{72} = v_{73} = 0$. Since V has Schmidt rank 3, the three blocks v_1, v_4, v_7 are pairwise linearly dependent. So V is locally equivalent to a matrix S the same as V , except that V_{11}, V_{14} are replaced by scalar matrices. It follows from $VV^\dagger = I$ that $r = d_B - r$. So S is locally equivalent to a matrix the same as S , except that the V_{i3} are replaced by scalar matrices. S is locally equivalent to a BCU which gives us a contradiction. So the case $\text{Dim } H = 2$ is excluded.

Let $\text{Dim } H = 3$. Up to a local unitary we may assume that H is spanned by v_2, v_3, v_6 . Since V has Schmidt rank 3, we have $v_1, v_4, v_5, v_7 \in H$. Hence $v_{i3} \propto v_{23}$ and $v_{i4} \propto v_{64}$. Since V is unitary and not a BCU, we have $v_{23} \neq 0$ and $v_{64} \neq 0$. Suppose v_{32}, v_{62} are linearly independent. It follows from $v_4, v_5 \in H$ that $v_4 = 0, v_{21} = v_{54} = 0$, and $v_{23} \propto v_{53}$. Since V is unitary, we have $r = d_B - r$, and hence $v_{13} = v_{14} = v_{71} = v_{73} = 0$. By $v_1 \in H$ we have $v_{64} = 0$ which gives us a contradiction. Hence $v_{32} \propto v_{62}$. Next suppose v_{21}, v_{31} are linearly independent. Since $v_5 \in H$, we have $v_{53} = v_{62} = 0$. Since $v_4 \in H$, we have $v_{44} \propto v_{64}$. So v_{64} is invertible. Because V is unitary we have $v_{72} = v_{73} = 0$. Since $v_7 \in H$, we have $v_{32} = 0$. It follows from $v_1 \in H$ that $v_{23} = 0$, which gives us a contradiction. So $v_{21} \propto v_{31}$. Since $v_i \in H$, we have $v_{i1} \propto v_{21}$ or $v_{i1} \propto v_{31}$, and $v_{i2} \propto v_{32}$ or $v_{i2} \propto v_{62}$. We may assume $v_{i1} = a_i A, v_{i2} = b_i B, v_{i3} = c_i C$, and $v_{i4} = d_i D$ with nonzero blocks A, B, C, D for $i = 1, \dots, 7$. Since we have proved $v_{23} \neq 0$ and $v_{64} \neq 0$, we have $c_2 d_6 \neq 0$. Since V

is unitary, we have

$$(|a_1|^2 + |a_2|^2)AA^\dagger + |b_1|^2BB^\dagger = I_{d_B-r}, \quad (5)$$

$$(|a_3|^2 + |a_4|^2)AA^\dagger + |b_3|^2BB^\dagger = I_{d_B-r}, \quad (6)$$

$$(a_1a_3^* + a_2a_4^*)AA^\dagger + b_1b_3^*BB^\dagger = 0, \quad (7)$$

$$(|c_1|^2 + |c_2|^2)CC^\dagger + |d_1|^2DD^\dagger = I_r, \quad (8)$$

$$|c_5|^2CC^\dagger + (|d_4|^2 + |d_5|^2)DD^\dagger = I_r, \quad (9)$$

$$|c_7|^2CC^\dagger + (|d_6|^2 + |d_7|^2)DD^\dagger = I_r, \quad (10)$$

$$c_5c_7^*CC^\dagger + (d_4d_6^* + d_5d_7^*)DD^\dagger = 0. \quad (11)$$

Since V is unitary and $c_2 \neq 0$, at least one of b_1, b_3 is nonzero. If one of them is zero, then (5) and (6) imply that A is proportional to a unitary. If neither of them is zero then (5) and (7) imply that A is proportional to a unitary. So we have proved A is always proportional to a unitary. It follows from (5) and (6) that BB^\dagger is proportional to I_{d_B-r} . Next, if one of c_5, c_7 is zero then (9) and (10) implies that D is proportional to a unitary. If both c_5, c_7 are nonzero, then (10) and (11) imply that D is proportional to a unitary. So we have proved D is always proportional to a unitary. It follows from $c_2 \neq 0$ and (8) that CC^\dagger is proportional to I_r . So V is locally equivalent to the following matrix

$$V' = \begin{pmatrix} a_1I_{d_B-r} & b_1B & a_2I_{d_B-r} & 0 & 0 & 0 \\ c_1C & d_1I_r & c_2C & 0 & 0 & 0 \\ a_3I_{d_B-r} & b_3B & a_4I_{d_B-r} & 0 & 0 & 0 \\ 0 & 0 & 0 & d_4I_r & c_5C & d_5I_r \\ 0 & 0 & 0 & b_6B & a_7I_{d_B-r} & b_7B \\ 0 & 0 & 0 & d_6I_r & c_7C & d_7I_r \end{pmatrix}, \quad (12)$$

where we still use the complex numbers a_i, b_i, c_i, d_i and blocks B, C since there is no confusion. Up to a global factor, we may assume that $BB^\dagger = I_{d_B-r}$. Since V' is unitary, we have

$$(|a_1|^2 + |a_2|^2)I_{d_B-r} + |b_1|^2BB^\dagger = I_{d_B-r}, \quad (13)$$

$$(|a_3|^2 + |a_4|^2)I_{d_B-r} + |b_3|^2BB^\dagger = I_{d_B-r}, \quad (14)$$

$$|d_1|^2I_r + (|b_1|^2 + |b_3|^2)B^\dagger B = I_r. \quad (15)$$

Recall that one of b_1, b_3 is nonzero. As $B^\dagger B$ and BB^\dagger have the same rank, from the three equations above we have $d_B - r = r$. Next we make B a diagonal matrix of nonnegative and real diagonal elements by doing singular value decomposition for B , but to preserve the identity blocks proportional to $I_{d_B-r} = I_r$ and I_r , the overall transform is of the form $V'' = (I_A \otimes (Q \oplus R))V'(I_A \otimes (Q^\dagger \oplus R^\dagger))$, where Q and R are $r \times r$ unitaries acting on subspaces of \mathcal{H}_B . So when V'' is expressed in the form of Eq. (12), we have $B = I_{d_B-r} = I_r$. Now C is also a $r \times r$ square matrix, and since CC^\dagger is proportional to I_r , C is normal, hence C is equivalent to a diagonal matrix under unitary similarity transform, so we can do the following transform on V'' to make C diagonal: $X = (I_A \otimes (S \oplus S))V''(I_A \otimes (S^\dagger \oplus S^\dagger))$, where S is a $r \times r$ unitary, and it turns out we still preserve $B = I_r$ due to the form of this transform. So V' is locally equivalent to a matrix X , which is still of the form (12) but B and C are replaced by diagonal matrices. So X is a BCU from the B side, and we have a contradiction. This completes the proof. \square

Let U be a bipartite unitary on $d_A \times d_B$ of Schmidt rank 3 and $d_A = 2, 3$. It follows from Theorem 3 and 6 that U is a controlled unitary. What's more, we can decide the side from which U is controlled by Lemma 2. To find out the explicit decomposition of U into a controlled unitary, we refer to an efficient algorithm constructed in [17] and references therein. The algorithm is proposed for finding the finest simultaneous singular value decomposition for simultaneous block-diagonalization of square matrices under unitary similarity.

4 Characterization of nonlocal unitary operators

In this section we propose a few applications of our results on general nonlocal unitary operators. First we characterize the equivalence of nonlocal unitaries and relate them to the controlled unitaries. In Theorem 7 we show that the SL -equivalent multipartite unitary operators are indeed locally equivalent. Using it and Theorem 6 we can simplify the problem of deciding the SL -equivalence of two bipartite unitaries of Schmidt rank 3 with $d_A = 2, 3$. Using Theorem 7 we provide a sufficient condition by which a bipartite unitary is locally equivalent to a controlled unitary in Corollary 8. Next we propose an upper bound on the quantum resources implementing bipartite unitaries of Schmidt rank 3 with $d_A = 2, 3$, see Lemma 9. We also show that this upper bound is saturated for some bipartite unitary. Third we apply our results to a special case of Conjecture 10 on the ranks of multipartite quantum states. This conjecture is to construct inequalities analogous to those in terms of von Neumann entropy such as the strong subadditivity [19].

4.1 Equivalence of nonlocal unitary operators

We start by presenting the following observation on the SL -equivalence of general nonlocal unitary operators.

Theorem 7 *Suppose U and V are multipartite unitaries acting on $\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_p$, and they satisfy $U = (S_1 \otimes S_2 \otimes \cdots \otimes S_p)V(T_1 \otimes T_2 \otimes \cdots \otimes T_p)$ for invertible operators S_i and T_i acting on $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_p$, respectively. Then $U = (Q_1 \otimes Q_2 \otimes \cdots \otimes Q_p)V(R_1 \otimes R_2 \otimes \cdots \otimes R_p)$, where Q_i and R_i are unitaries acting on $\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_p$, respectively. In particular, when S_i and T_i are identity operators on any party i , we can choose Q_i and R_i to be identity operators.*

Proof. Suppose S_i and T_i have singular value decompositions of the form $S_i = E_i C_i F_i$, $T_i = G_i D_i H_i$, where E_i, F_i, G_i and H_i are unitaries, C_i and D_i are real diagonal matrices with the diagonal elements sorted in non-increasing order. The diagonal elements of C_i and D_i are the singular values of S_i and T_i , respectively. Since S_i and T_i are invertible, all the diagonal elements of C_i and D_i are positive.

Let $U' = (E_1^\dagger \otimes E_2^\dagger \otimes \cdots \otimes E_p^\dagger)U(H_1^\dagger \otimes H_2^\dagger \otimes \cdots \otimes H_p^\dagger)$, and let $V' = (F_1 \otimes F_2 \otimes \cdots \otimes F_p)V(G_1 \otimes G_2 \otimes \cdots \otimes G_p)$, then U' and V' are unitaries and satisfy

$$U' = (C_1 \otimes C_2 \otimes \cdots \otimes C_p)V'(D_1 \otimes D_2 \otimes \cdots \otimes D_p). \quad (16)$$

Using $U'U'^\dagger = I$, where I is the identity operator on the entire space, we have

$$I = U'U'^\dagger = (C_1 \otimes C_2 \otimes \cdots \otimes C_p)V'(D_1^2 \otimes D_2^2 \otimes \cdots \otimes D_p^2)V'^\dagger(C_1 \otimes C_2 \otimes \cdots \otimes C_p), \quad (17)$$

and using $V'^\dagger = V'^{-1}$, we get

$$V' = (C_1^2 \otimes C_2^2 \otimes \cdots \otimes C_p^2)V'(D_1^2 \otimes D_2^2 \otimes \cdots \otimes D_p^2). \quad (18)$$

And since C_i and D_i are diagonal, $\tilde{C} := C_1^2 \otimes C_2^2 \otimes \cdots \otimes C_p^2$ and $\tilde{D} := D_1^2 \otimes D_2^2 \otimes \cdots \otimes D_p^2$ are diagonal. Consider any nonzero element in the matrix V' , and let us suppose it is on row j and column k of V' . Then Eq. (18) implies $\tilde{C}_{jj}\tilde{D}_{kk} = 1$, where \tilde{C}_{jj} means the j -th diagonal element of \tilde{C} , and \tilde{D}_{kk} is similarly defined. And since \tilde{C} and \tilde{D} only contain positive elements on their diagonals, we have $\sqrt{\tilde{C}_{jj}}\sqrt{\tilde{D}_{kk}} = 1$. This holds for any 2-tuple (j, k) satisfying that the element on row j and column k of V' is nonzero, and since $C_1 \otimes C_2 \otimes \cdots \otimes C_p$ and $D_1 \otimes D_2 \otimes \cdots \otimes D_p$ are diagonal, this implies

$$V' = (C_1 \otimes C_2 \otimes \cdots \otimes C_p)V'(D_1 \otimes D_2 \otimes \cdots \otimes D_p). \quad (19)$$

Together with Eq. (16), we get $U' = V'$, hence

$$U = (E_1 F_1 \otimes E_2 F_2 \otimes \cdots \otimes E_p F_p) V (G_1 H_1 \otimes G_2 H_2 \otimes \cdots \otimes G_p H_p), \quad (20)$$

where $E_i F_i$ and $G_i H_i$ are unitaries by construction. From the proof above we see that when S_i and T_i are identity operators on any party i , we can choose E_i , F_i , G_i and H_i to be identity operators. This completes the proof of Theorem 7. \square

The theorem implies that two SL-equivalent multipartite unitary operators are indeed locally equivalent to each other. Such two unitaries can be viewed as the same nonlocal resource in quantum information processing tasks. In contrast, two stochastic LOCC (SLOCC)-equivalent pure states may be not locally equivalent, and generally they can only probabilistically simulate each other in quantum information processing tasks. For example, the 3-qubit W state $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$ [20] and W-like state $|W'\rangle = \frac{1}{2}|001\rangle + \frac{1}{2}|010\rangle + \frac{1}{\sqrt{2}}|100\rangle$ are SLOCC-equivalent but not locally equivalent, as the bipartition of them give rise to a non-maximally entangled state and a maximally entangled state, respectively.

It is known that the classification of multipartite states under LOCC and SLOCC are different, because they are realized with probability one and less than one, respectively. So the former is more coarse-grained than the latter. For example, the three-qubit pure states have infinitely many orbits under LOCC [21], while there are only two kinds of fully entangled states under SLOCC, namely the GHZ and W states [20]. In contrast, Theorem 7 implies that the classification of multipartite unitary operations under local unitaries and SL are essentially the same, the latter does not give any additional advantage the former does not have. There are other ways of classifying nonlocal unitaries, such as the LO, LOCC, SLOCC equivalences discussed in [6], which implicitly assume the use of ancillas.

Based on the previous results we can simplify the decision of SL-equivalence of two bipartite unitaries U, V of Schmidt rank 3 and $d_A = 2, 3$. In practice this is motivated by the simulation of one of them by the other, and the implementation of them. Using Theorem 7 we only need to study the equivalence under local unitaries. It follows from Theorem 6 that both U, V are controlled unitaries. They are not locally equivalent if they are not controlled from the same side, which can be decided by the algorithm in [17]. Nevertheless, deciding the equivalence of two controlled unitary controlled from the same side remains unknown.

Below we characterize the controlled unitaries using Theorem 7.

Corollary 8 *If a unitary U on $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is SL-equivalent to*

$$V = \sum_{j=1}^m R_j \otimes V_j \quad (21)$$

where R_j are operators on \mathcal{H}_A satisfying

$$P_j R_j P_j = R_j, \quad \forall j, \quad (22)$$

with $\{P_j\}$ being a set of mutually orthogonal projectors on \mathcal{H}_A , and V_j are arbitrary operators on \mathcal{H}_B . Then U is equivalent under local unitaries to the block diagonal form

$$U' = \sum_{j=1}^m P_j \otimes V'_j, \quad (23)$$

where V'_j are unitary operators on \mathcal{H}_B .

In particular, if a unitary U on \mathcal{H} is SL-equivalent to $\sum_{j=1}^{d_A} |j\rangle\langle j| \otimes U_j$ for nonzero matrices U_j , then U is a controlled unitary gate controlled from the A side.

Proof. Note that the general case is reducible to the particular case by first doing singular value decompositions of R_j , and at the end noting that the final local unitaries V_j' on \mathcal{H}_B corresponding to the same R_j are the same. Hence we only need to prove the particular case.

By hypothesis, U is locally equivalent to $V = \sum_{j=1}^{d_A} |\alpha_j\rangle\langle\beta_j| \otimes V_j$. The states $|\alpha_1\rangle, \dots, |\alpha_{d_A}\rangle \in \mathcal{H}_A$ are linearly independent, and the states $|\beta_1\rangle, \dots, |\beta_{d_A}\rangle \in \mathcal{H}_B$ are also linearly independent. Let $|\gamma\rangle \perp P$, and $P = I_A - |\gamma\rangle\langle\gamma|$ the projector on the hyperplane of \mathcal{H}_1 spanned by $|\alpha_2\rangle, \dots, |\alpha_{d_A}\rangle$. Since V is unitary, we have $\langle\gamma|_A V V^\dagger |\gamma\rangle_A = |\langle\gamma|\alpha_1\rangle|^2 V_1 V_1^\dagger = I_B$. So the matrix V_1 is proportional to a unitary matrix. We may replace $|\alpha_2\rangle, \dots, |\alpha_{d_A}\rangle$ by any $d_A - 1$ states of $|\alpha_1\rangle, \dots, |\alpha_{d_A}\rangle$ in the above argument, and similarly obtain that the V_i 's are proportional to unitary matrices, $i = 2, \dots, d_A$. So U is SL-equivalent to a controlled unitary from the A side. The assertion then follows from Theorem 7. This completes the proof. \square

An explanation of Corollary 8 is as follows: if the effect of a unitary is to stochastically implement a controlled type operation of the form in Eq. (21), then the unitary must be a controlled unitary.

4.2 Entanglement cost of implementing a bipartite unitary

Computing the entanglement cost of implementing a nonlocal unitary is an important question in quantum information [6]. For this purpose a few protocols have been constructed. For example, one can use teleportation [22] twice to implement a nonlocal unitary by using LOCC and two maximally entangled states $|\Psi_{d_A}\rangle$ ($d_A \leq d_B$), which contains $2 \log_2 d_A$ ebits [8]: Alice teleports her input system to Bob, and Bob does the unitary locally, and teleports back the part of the output system belonging to Alice to her. In ref. [8], another protocol has been proposed to implement any bipartite controlled unitary controlled from A side by LOCC and the maximally entangled state $|\Psi_{d_A}\rangle$. Using these protocols, and Theorem 3 and 6, we have

Lemma 9 *Let $d_A = 2, 3 \leq d_B$. Any bipartite unitary of Schmidt rank 3 can be implemented by using LOCC and the maximally entangled state $|\Psi_k\rangle$, where $k = \min\{d_A^2, d_B\}$.*

From this lemma, $\log_2 d_B$ ebits is an upper bound of the amount of entanglement needed to implement a bipartite unitary of Schmidt rank 3. In the following we show that this upper bound can be saturated for some unitary with $d_A = 2, d_B = 3$. Let $U = I_2 \otimes |1\rangle\langle 1| + \sigma_x \otimes |2\rangle\langle 2| + \sigma_y \otimes |3\rangle\langle 3|$ be on the space $\mathcal{H}_A \otimes \mathcal{H}_B$, A' be the ancilla qubit system, and the bipartite space $\mathcal{K} = \mathcal{H}_{AA'} \otimes \mathcal{H}_B$. Let the product state $|\psi\rangle = \frac{1}{\sqrt{2}}(|11\rangle + |22\rangle) \otimes \frac{1}{\sqrt{3}}(|1\rangle + |2\rangle + |3\rangle) \in \mathcal{K}$. Then $U|\psi\rangle \in \mathcal{K}$ is a uniformly entangled state of Schmidt rank 3. That is, U creates $\log_2 3$ ebits and therefore implementing U must cost at least so much entanglement [7]. On the other hand, Lemma 9 implies that U can be implemented using $\log_2 3$ ebits and LOCC. We leave as an open question whether there is a Schmidt-rank-3 unitary with $d_A = 2, d_B = 4$ that needs 2 ebits to implement using LOCC.

4.3 A conjecture for the ranks of quantum states

The following conjecture is proposed in [19]. In the following T denotes the matrix transpose.

Conjecture 10 *Let R_1, \dots, R_K be $m_1 \times n_1$ complex matrices, and let S_1, \dots, S_K be $m_2 \times n_2$ complex matrices. Then*

$$\text{rank} \left(\sum_{i=1}^K R_i \otimes S_i^T \right) \leq K \cdot \text{rank} \left(\sum_{i=1}^K R_i \otimes S_i \right). \quad (24)$$

Note that $\text{rank}(\sum_{i=1}^K R_i^T \otimes S_i) = \text{rank}(\sum_{i=1}^K R_i \otimes S_i^T)$ holds generally. The motivation of this conjecture is to construct basic inequalities in terms of ranks of multipartite quantum states, and

some of them have been constructed in [19]. They are analogous to the inequalities in terms of von Neumann entropy such as the strong subadditivity. Using the basic inequalities one can constrain the relation of the ranks of different marginals and quantify the multipartite entanglement dimensionality.

The conjecture with $K = 1$ is trivial, as the transpose does not change the rank of a matrix. Next Conjecture 10 with $K = 2$ has been proved in [19]. However the conjecture with $K \geq 3$ is still an open problem and is considered to be highly nontrivial in matrix theory. Nevertheless, the results in last section shed some light on the conjecture with $K = 3$. Let $U = \sum_{i=1}^3 R_i \otimes S_i$ be a $3 \times d_B$ unitary matrix. Let $U^\Gamma = \sum_{i=1}^3 R_i \otimes S_i^T$ be the partial transpose of U [23] with the B side transposed. If U is of Schmidt rank 3, Theorems 3 and 6 imply that U is locally equivalent to a controlled unitary; if the Schmidt rank of U is less than 3, U is also locally equivalent to a controlled unitary, according to [10]. The controlled unitary could be controlled from either side, and in either case we have $\text{rank } U^\Gamma = \text{rank } U$. Hence $\text{rank } U^\Gamma \leq 3 \cdot \text{rank } U$, which is Conjecture 10 with $K = 3$. Evidently, if Theorem 6 can be generalized to any $d_A > 3$, Conjecture 10 would hold for all Schmidt-rank-3 unitaries $U = \sum_{i=1}^3 R_i \otimes S_i$.

5 Conclusions

We have shown that the nonlocal unitary operator of Schmidt rank 3 on the $d_A \times d_B$ system is locally equivalent to a controlled unitary when $d_A \leq 3$. Using this result we have shown that LOCC and the $r \times r$ maximally entangled state of $r = \min\{d_A^2, d_B\}$ are sufficient to implement such operators. We also have shown that SL-equivalent nonlocal unitary operators are indeed locally equivalent. In addition we have verified a special case of Conjecture 10 on the ranks of multipartite quantum states, when the argument in the bracket of (24) is a bipartite unitary of Schmidt rank 3 and $d_A \leq 3$.

Unfortunately we are not able to prove Conjecture 1 when $d_A > 3$, as the proof of Theorem 6 cannot be easily generalized. We believe that the generalization of this theorem will prove Conjecture 1 and verify more cases of Conjecture 10. Otherwise, the first counterexample to Conjecture 1 might exist when $d_A = d_B = 4$. The next interesting question is whether a BCU from B side of Schmidt rank 3 is a controlled unitary. This would generalize Lemma 4. Third apart from the Schmidt rank, is there another physical quantity which describes the local equivalence between a nonlocal unitary and a controlled unitary? This is to investigate the connection between nonlocal and controlled unitaries of arbitrary Schmidt rank.

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